Existence of a fixed point of a nonsmooth function arising in numerical mechanics

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This paper is dedicated to Jean-Baptiste Hiriart-Urruty on the occasion of his 60^{th} birthday. Merci Jean-Baptiste pour ton soutien constant et pour nous avoir fait partager tes connaissances en analyse variationnelle - et ton sens de l'humour !

Abstract

A recent work [ACML10] introduces a formulation as a nonsmooth fixed-point problem of a basic problem in numerical mechanics (namely the dynamical Coulomb friction problem in finite dimension with discretized time). Using this new formulation, the existence of a solution to the problem and its numerical resolution are then guaranteed under a strong assumption on the data of this problem.

In this paper, we show that the fixed point problem admits solution under a natural, weaker assumption. This existence proof uses a perturbation argument combined with continuity properties of a set-valued mapping associated with the constraints of the problem.

Contents

1 Introduction, motivation, notation

1.1 Presentation of the problem

As shown by the seminal work of Jean-Jacques Moreau, nonsmooth analysis and mechanics have nice interplays. For instance, contact mechanics make a fundamental use of nonsmooth objects for modeling and numerical simulation, as for example convex cones to express friction. The recent paper [ACML10] focuses on the numerical problem arising when discretizing the dynamics of mechanical system with contact and friction. It formulates the incremental problem as a nonsmooth fixed-point problem, and uses this new formulation to get a basic result of existence of solutions together with a new way to compute them. The sections 1.3 and 1.4 sketch the context of numerical mechanics and briefly review the existence result and its consequences.

In this paper, we show that there exists a solution to the above fixed-point theorem under a natural assumption - weaker that the assumption used in [ACML10]. The proof of the existence of a fixed point under the weak assumtion relies on the application of the standard Brouwer fixed-point theorem, but to get the boundeness and continuity properties of the function we use nonstandard arguments from set-valued analysis and sensitivity analysis in optimization.

The function that we study in this paper is defined by the forthcoming (1.19) using the solution of a conic optimization problem. The remainder of this introduction presents the basic definitions of second-order conic optimization (Subsection 1.2), details the notation by sketching the mechanical context (Subsection 1.3), then recalls the existing results and precises the goal of this paper (Subsection 1.4).

1.2 Second-order cone programming

The function F we consider in this paper is defined using the solution of an optimization problem with second-order cone constraints (see forthcoming (1.14)). Second-order cone constraints indeed appear naturally in mechanics in the Coulomb friction law (see Section 1.3). Here we only introduce the notation that we need; and we refer to [AG03] and [BV04] for more on second-order cone programming.

Given a vector $x \in \mathbb{R}^d$, the subscripts "N" and "T" indicate normal and tangential components of a vector with respect to a given unit vector $e \in \mathbb{R}^d$. In other words,

$$x_N := x^\top e \in \mathbb{R} \quad \text{and} \quad x_T := x - x_N e \in \mathbb{R}^d. \tag{1.1}$$

The so-called second-order cone $K_{e,\mu}$ directed by the unit vector $e \in \mathbb{R}^k$ and of parameter $\mu \in [0, +\infty[$ is defined by the closed and convex cone

$$K_{e,\mu} := \left\{x \in \mathbb{R}^d \ : \ \|x_T\| \leq \mu x_N\right\}.$$

Note the two extreme cases

$$K_{e,0} := \left\{x \in \mathbb{R}^d \ : \ x_T = 0,\ x_N \geq 0\right\} \quad \text{and by definition} \quad K_{e,\infty} := \left\{x \in \mathbb{R}^d \ : \ x_N \geq 0\right\}.$$

It is easy to see that the dual cone of the second-order cone $K_{e,\mu}$ (with $\mu \in]0, \infty[$) is also a second-order cone:

$$K^*_{e,\mu} := \left\{s \in \mathbb{R}^d : \ x^\top s \geq 0 \text{ for all } x \in K_{e,\mu}\right\} = K_{e,\frac{1}{\mu}}.$$

This also holds for $\mu = 0$ and $\mu = \infty$, with the convention that $1/0 = \infty$ and $1/\infty = 0$: we have indeed $(K_{e,0})^* = K_{e,\infty}$ and $(K_{e,\infty})^* = K_{e,0}$. For given n unit vectors and n scalars

$$e^1, \ldots, e^n \in \mathbb{R}^d \qquad \text{and} \qquad \mu_1, \ldots, \mu_n \in [0, +\infty], \tag{1.2}$$

we introduce the associated product-cone

$$L := K_{e^1,\mu^1} \times \cdots \times K_{e^n,\mu^n} \;\subset\; \mathbb{R}^{nd}, \tag{1.3}$$

whose dual cone is

$$L^* = K^*_{e^1,\mu^1} \times \cdots \times K^*_{e^n,\mu^n} = K_{e^1,\frac{1}{\mu^1}} \times \cdots \times K_{e^n,\frac{1}{\mu^n}}. \tag{1.4}$$

We will also consider (in (1.21) below) another convex cone of the same form as above: given the data (1.2), we consider

$$I := \Big\{ i \in \{1,\ldots,n\} : \mu^i \neq 0 \Big\} \qquad \text{and} \qquad n_I := \operatorname{Card} I, \tag{1.5}$$

and we set for all $i = 1,\ldots,n$

$$\mathcal{K}^{*i} := \begin{cases} \operatorname{int} K^*_{e^i,\mu^i} = \operatorname{int} K_{e^i,\frac{1}{\mu^i}} & \text{if } i \in I \\ K_{e^i,\infty} & \text{if } i \notin I \end{cases}$$

We can then introduce similarly to (1.4) the convex cone (included in L^*)

$$\mathcal{L}^* = \mathcal{K}^{*1} \times \cdots \times \mathcal{K}^{*n} \subset \mathbb{R}^{nd}, \tag{1.6}$$

that appears in the (strong) assumption of Theorem 3.4. Let us just mention that it is not a dual cone (it is not closed), yet we denote it with a star, due to its ressemblance with L^*.

1.3 Mechanical context

This section briefly presents the context in numerical mechanics where appears the fixed point problem that we consider in this paper. We refer to the introduction of [ACML10] for references and more details. This section can be skipped in a first reading.

Simulating the dynamics of mechanical systems which involve unilateral contact between their parts or with external objects is common in engineering (granular materials, robotics, computer graphics,...), and have been extensively studied by the community of contacts mechanics. One difficulty is to handle the nonregularity due to the friction between objects. The recent work [ACML10] (see also [Cad09]) proposes a new approach by convex optimization and fixed point. We sketch here the problem, its mathematical formulation and set the notation for the rest of the paper.

Consider a mechanical system in $\mathbb{R}^d$ (in practice, $d = 2$ or $d = 3$) with several bodies having n contacts and m degrees of freedom. Discretizing the dynamics of a mechanical system with contact and friction has the following standard modelisation. A superscript $i \in \{1,\ldots,n\}$ corresponds to a contact between two of the bodies of the system: the vector $e^i \in \mathbb{R}^d$ gives the normal direction of the contact, $\tilde{u} := (\tilde{u}^1,\ldots,\tilde{u}^n) \in \mathbb{R}^{nd}$ are the (tilted) relative velocities at contact points, and $r := (r^1,\ldots,r^n) \in \mathbb{R}^{nd}$ the discretized impulses. The nonregularity of the problem comes from Coulomb friction law at i which expresses, for the friction coefficient at contact $i\mu^i \in [0,\infty[$, that the couple $(\tilde{u}^i, r^i)$ satisfies

$$(K_{e^i,\mu^i})^* \ni \tilde{u}^i \perp r^i \in K_{e^i,\mu^i}. \tag{1.7}$$

The other relations between the variables are supposed to be affine: the generalized velocities $v \in \mathbb{R}^m$ are connected to the impulses r by a dynamical relation (see forthcoming (1.8)), and to the relative velocities $\tilde{u}$ by a kinematical equation (see (1.9)). To express them, we consider an additional tilting variable $s_i \in \mathbb{R}$ at each contact with friction ($\mu^i \neq 0$); that is $s \in \mathbb{R}^{n_I}$, with n_I defined by (1.5). The formulation of the incremental discretized problem ends up with the following conic complementarity problem (whose data is detailed right after) with respect to the variable $(v, r, \tilde{u}, s) \in \mathbb{R}^m \times \mathbb{R}^{nd} \times \mathbb{R}^{nd} \times \mathbb{R}^{n_I}$

$$Mv + f = H^\top r \tag{1.8}$$

$$\tilde{u} = Hv + w + Es \tag{1.9}$$

$$L^* \ni \tilde{u} \perp r \in L \tag{1.10}$$

$$s^i = \|\tilde{u}^i_T\| \quad \text{for } i \in I \tag{1.11}$$

with L and L^* defined by (1.3) and (1.4) respectively. Thus the data of the problem is

$$e^i \in \mathbb{R}^d,\ \mu^i \in [0, +\infty],\ M \in \mathbb{R}^{m\times m},\ f \in \mathbb{R}^m,\ H \in \mathbb{R}^{nd\times m},\ w \in \mathbb{R}^{nd} \quad \text{and} \quad E \in \mathbb{R}^{nd\times n_I}. \tag{1.12}$$

The mass matrix $M \in \mathbb{R}^{m\times m}$ is assumed definite positive, but H, w and f have no properties. In contrast, the matrix $E \in \mathbb{R}^{nd\times n_I}$ has very special structure: it is constructed by concatenating n_I columns $E_i \in \mathbb{R}^{nd}$, where E_i is itself the concatenation of n vectors of $\mathbb{R}^d$, all zeros except for the i-th which is $\mu^i e^i$. Here is an example to fix ideas: for $d = 2$, $n = 3$, $e^1 = e^2 = e^3 = [0; 1]$, $\mu^1 = 1$, $\mu^2 = 0$, $\mu^3 = 2$, the matrix E is

$$E^\top = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The construction of E gives the following property that will be useful to establish our result: for any s and t in $\mathbb{R}^{n_I}$ and $i = 1, \ldots, n$,

$$(Hv + w + Et)^i = \begin{cases} (Hv + w + Es)^i & \text{if } i \notin I \\ (Hv + w + Es)^i + \mu^i(t^i - s^i)e^i & \text{if } i \in I. \end{cases} \tag{1.13}$$

1.4 Fixed point problem and previous existence result

The function that we study in this paper is defined with the solution of the following quadratic second-order cone optimization problem parameterized by $s \in \mathbb{R}^{n_I}_+$

$$\begin{cases} \min & \frac{1}{2}v^\top Mv + f^\top v \\ Hv + w + Es \in L^* \end{cases} \tag{1.14}$$

with the data of (1.12). The quadratic objective function

$$J(v) := \frac{1}{2}v^\top Mv + f^\top v \tag{1.15}$$

is strongly convex and inf-compact (since M is assumed to be positive definite), so that, whenever its closed convex feasible set

$$\bar{C}(s) := \{v \in \mathbb{R}^m : Hv + w + Es \in L^*\} \tag{1.16}$$

is nonempty, (1.14) has a unique solution, that we call $v(s)$. In other words, assuming that $\bar{C}(s) \neq \emptyset$, we introduce

$$v(s) := \underset{v\in\bar{C}(s)}{\operatorname{argmin}}\, J(v) \in \mathbb{R}^m\,. \tag{1.17}$$

As stated in the next theorem, this defines a function $v\colon \mathbb{R}_+^{n_I} \to \mathbb{R}_+$, which in turn yields two more mappings

$$\mathbb{R}_+^{n_I} \ni s \quad \longmapsto \quad \tilde{u}(s) := Hv(s) + w + Es \in \mathbb{R}^{nd} \tag{1.18}$$

$$\mathbb{R}_+^{n_I} \ni s \quad \longmapsto \quad F(s) := (\|\tilde{u}_T^1(s)\|, \ldots, \|\tilde{u}_T^n(s)\|) \in \mathbb{R}^{nd}\,. \tag{1.19}$$

We have more precisely the following result, that also connects the above-defined F with the mechanical problem of the previous section.

Theorem 1.1 (Definition of F and connection with mechanical problem)**.** *Suppose*

$$\exists\, v \in \mathbb{R}^m \quad \text{such that } Hv + w \in L^* \tag{1.20}$$

(that is $\bar{C}(0) \neq \emptyset$) then $\bar{C}(s) \neq \emptyset$ for all $s \in \mathbb{R}_+^{n_I}$, so that the function F is well-defined on $\mathbb{R}_+^{n_I}$ by (1.19)*. Moreover, if $(v^*, r^*, \tilde{u}^*, s^*)$ solve the system* (1.8)–(1.11)*, then $v^* = v(s^*)$ and s^* is a fixed point of F*

$$F(s^*) = s^*.$$

This result suggests a new approach to solve the mechanical system. Essentially the idea is to isolate the convexity in (1.8)-(1.11) and to treat it by optimization. In practice we compute fixed-point iterations on F after solving (1.14) and check if it is a solution of the system. Numerical experiments of [Cad09] and [ACML10] shows that this gives a simple, cheap and surprisingly robust way to tackle this problem.

This approach indeed works very often - but it may fail for two reasons: either the solution does not exist, or the algorithm does not find it for some numerical reasons. Simple examples show indeed that F may have no fixed point and the system no solution as well (see the adapted Painlevé counter-example in [ACML10]). So this leads to the question of the existence of fixed-point, that was solved with the celebrated Brouwer Theorem, under a strong assumption (involving (1.6)).

Theorem 1.2 (Continuity and existence)**.** *If there exists*

$$v \in \mathbb{R}^m \quad \text{such that } Hv + w \in \mathcal{L}^* \tag{1.21}$$

then the function F defined by (1.19) *is continuous, as well as bounded, on $\mathbb{R}_+^{n_I}$. Thus we have the existence of a fixed-point to F.*

Obviously, (1.20) is weaker than (1.21). The goal of this paper is then to prove that Theorem 1.2 is still valid under the weaker assumption (1.20). This will be done in Theorem 3.5 and this turns out to rely two ingredients

- some continuity properties of the constraint-set of (1.14) (see Section 2), and
- a perturbation argument (see Section 3.2).

2 Continuity properties of the constraint-set mapping

This section gathers the continuity properties of the multi-application $\bar{C}\colon \mathbb{R}_+^{n_I} \rightrightarrows \mathbb{R}^m$ defined by (1.16) (that we also bound, see forthcoming (2.3)). We start with a few definitions and an easy basic property regarding continuity for multi-applications (for more details, see [HUL93, Appendix] or [RW98]).

2.1 Generalities on continuity of set-valued mappings

The distance of a point x to a closed convex set S is defined by $d(x, S) := \min_{s\in S} \|x - s\|$, the excess of a set S_1 over a set S_2 by

$$e_H(S_1/S_2) := \sup\{d(x, S_2),\ x \in S_1\}$$

and the Hausdorff distance between S_1 and S_2 by

$$\Delta_H(S_1, S_2) := \max(e_H(S_1/S_2), e_H(S_2/S_1)).$$

A multi-application $S\colon D \subset \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is said to be outer semi-continuous at $\bar{s}$ when for all $\varepsilon > 0$, there exists a neighborhood N of $\bar{s}$ such that for all $s \in N$

$$S(s) \subset S(\bar{s}) + B(0, \varepsilon) \qquad \text{or, in other words,} \qquad e_H(S(s)/S(\bar{s})) \leq \varepsilon.$$

Similarly, S is said to be inner semi-continuous at $\bar{s}$ when for all $\varepsilon > 0$, there exists a neighborhood N of $\bar{s}$ such that for all $s \in N$

$$S(\bar{s}) \subset S(s) + B(0, \varepsilon) \qquad \text{or, in other words,} \qquad e_H(S(\bar{s})/S(s)) \leq \varepsilon.$$

Moreover S is said to be continuous at $\bar{s}$ when it is both outer and inner semi-continuous at $\bar{s}$. Finally S is said to be closed when its graph is closed, that is to say

$$\forall (s_k)_k \in D \text{ with } s_k \to \bar{s},\ \forall (v_k)_k \in S(s_k) \text{ with } v_k \to \bar{v},\ \bar{s} \in D \text{ and } \bar{v} \in S(\bar{s}), \tag{2.1}$$

and *bounded* when $S(D)$ is bounded.

Let us start with an easy general result which generalizes the following lemma: if a (single-valued) function is continuous over a compact set, then it is uniformly continuous over this set.

Lemma 2.1. *Let S be a closed, inner semi-continuous and bounded multi-application defined on a compact set C. Then S is uniformly continuous on C:*

$$\forall \varepsilon > 0, \exists \delta > 0,\ \forall x, y \in C,\ \|x - y\| \leq \delta \ \Rightarrow\ \Delta_H(S(x), S(y)) \leq \varepsilon.$$

Proof. To prove this result by contradiction, assume that there exists $\varepsilon > 0$ such that for all $k = 1, 2\ldots$ with $\delta_k \to 0$ as $k \to \infty$, there exist x_k and y_k with $\|x_k - y_k\| \leq \delta_k$ and $\Delta_H(S(x_k), S(y_k)) > \varepsilon$. This means that, for all k: either there exists $u_k \in S(x_k)$ such that $d(u_k, S(y_k)) > \varepsilon$; or that there exists $v_k \in S(y_k)$ such that $d(v_k, S(x_k)) > \varepsilon$. At least one of the two sequences u_k and v_k (say u_k) is infinite, and up to re-numbering the sequence one may assume that it is defined for all $k \in \mathbb{N}$. Hence we have

$$d(u_k, S(y_k)) > \varepsilon \quad \text{for all } k. \tag{2.2}$$

Since C is compact and S is bounded (hence u_k is bounded), we can assume (up to extraction of a subsequence) that $x_k \to \ell \in C$ and $u_k \to \bar{u}$. The multi-application S being closed by assumption, we have $\bar{u} \in S(\ell)$. Since $\|x_k - y_k\| \to 0$, we also have $y_k \to \ell$. For k large enough, we have $d(\bar{u}, S(y_k)) \leq \varepsilon/3$ (by inner semi-continuity) and we also have $\|u_k - \bar{u}\| \leq \varepsilon/3$ (since $u_k \to \bar{u}$). Then there holds

$$d(u_k, S(y_k)) \leq d(u_k, \bar{u}) + d(\bar{u}, S(y_k)) \leq \frac{2\varepsilon}{3} < \varepsilon$$

which is a contradiction with (2.2) and ends the proof. ■

Note that a closed and bounded multi-application is necessarily outer semi-continuous, as one easily shows, therefore the multifunction S in the previous lemma is continuous.

2.2 The bounded constraint-set mapping

Let us now focus on $\bar{C}(\cdot)$, more precisely on its bounded counterpart

$$G \colon \begin{cases} \mathbb{R}_+^{n_I} \longrightarrow K \\ \quad s \quad \longmapsto \bar{C}(s) \cap K, \end{cases} \tag{2.3}$$

with

$$K := \{v \in \mathbb{R}^m \ : \ J(v) \leq J(v(0))\}. \tag{2.4}$$

Note that K is a convex and compact set, as the sublevel set of the strongly convex quadratic function J.

Theorem 1.1 states in particular that the set-valued functions $\bar{C}$, and thus G, have nonempty ranges on $\mathbb{R}_+^{n_I}$. Two easy properties of G are the following.

Lemma 2.2. *The graph of the multi-application G defined by* (2.3) *is closed and convex.*

Proof. Let $(s_1, v_1), (s_2, v_2) \in \operatorname{graph}(G)$ and $\alpha \in [0, 1]$. We have $u_1 := Hv_1 + w + Es_1 \in L^*$ and $u_2 := Hv_2 + w + Es_2 \in L^*$. The convexity of L^* implies $\alpha u_1 + (1 - \alpha)u_2 \in L^*$. Said otherwise,

$$H(\alpha v_1 + (1 - \alpha)\, v_2) + w + E(\alpha s_1 + (1 - \alpha)\, s_2) \in L^*,$$

so that $(\alpha v_1 + (1 - \alpha)\, v_2) \in \bar{C}(\alpha s_1 + (1 - \alpha)\, s_2)$. In addition, the convexity of K implies $(\alpha v_1 + (1 - \alpha)\, v_2) \in K$, so that $\alpha(s_1, v_1) + (1 - \alpha)(s_2, v_2) \in \operatorname{graph}(G)$: therefore $\operatorname{graph}(G)$ is convex.

Let $s_k \in \mathbb{R}_+^{n_I}$ with $s_k \to \bar{s}$ and $v_k \in G(s_k)$ with $v_k \to \bar{v}$. By definition there holds $Hv_k + w + Es_k \in L^*$; using the fact that H is continuous and L^* is closed, there holds $H\bar{v} + w + E\bar{s} \in L^*$. Moreover K is compact, hence $\bar{v} \in K$. All this gives $\bar{v} \in G(\bar{s})$, therefore G is closed. ∎

Lemma 2.3 (Monotonicity)**.** *The multi-applications $\bar{C}$ and G are increasing; in other words for $s, t \in \mathbb{R}^{n_I}$ such that $s^i \leq t^i$ for all i, we have $\bar{C}(s) \subset \bar{C}(t)$ and $G(s) \subset G(t)$.*

Proof. Let $s, t \in \mathbb{R}^{n_I}$ such that $s^i \leq t^i$ and take $v \in \bar{C}(s)$, i.e. $(Hv + w + Es)^i \in K^*_{e^i,\mu^i}$ for $i = 1, \ldots, n$. Let us show componentwise that we also have $Hv + w + Et \in L^*$. We see from (1.13) that $(Hv + w + Et)^i \in K^*_{e^i,\mu^i}$ if $i \notin I$. On the other hand, take $i \in I$; knowing that $e^i \in K^*_{e^i,\mu^i}$ and $\mu^i(t^i - s^i) \geq 0$,

$$\mu^i(t^i - s^i)e^i \in K^*_{e^i,\mu^i}.$$

Since $K^*_{e^i,\mu^i}$ is convex

$$z := \frac{1}{2}(Hv + w + Es)^i + \frac{1}{2}\mu^i(t^i - s^i)e^i \in K^*_{e^i,\mu^i}.$$

By positive homogeneity, $(Hv + w + Et)^i = 2z$ also lies in $K^*_{e^i,\mu^i}$. ∎

Let us now turn to the more elaborate property of inner semi-continuity of G.

Lemma 2.4. *The multi-application G defined by* (2.3) *is inner semi-continuous on $\mathbb{R}_+^{n_I}$.*

Proof. Let $\bar{s} \in \mathbb{R}_+^{n_I}$ and $\varepsilon > 0$. It suffices to show that

$$\exists \delta > 0 \ : \ \forall s \in \mathbb{R}_+^{n_I}, \quad \|s - \bar{s}\|_\infty \leq \delta \ \Rightarrow \ G(\bar{s}) \subset G(s) + B(0, \varepsilon).$$

If $\bar{s} = 0$, this is obvious since G is increasing (Lemma 2.3). Otherwise, let $\chi := \min_i\{\bar{s}^i : \bar{s}^i > 0\} > 0$. Let also $\bar{v} \in G(\bar{s}) \neq \emptyset$, and let $v_0 \in G(0) \neq \emptyset$; we may assume that $v_0 \neq \bar{v}$, otherwise $\bar{v} \in G(0) \subset G(s)$ for all $s \geq 0$ and there is nothing to prove. We will show that

$$\delta := \min\left\{\chi, \frac{\chi\varepsilon}{\|v_0 - \bar{v}\|}\right\} > 0$$

does the job; note that $\bar{s}_i - \delta \geq 0$ for all i such that $\bar{s}_i > 0$. Consider now the following convex combination

$$s_\alpha := (1 - \alpha)\, 0 + \alpha\, \bar{s} = \alpha\, \bar{s}$$

of $0 \in \mathbb{R}^m$ and $\bar{s}$, where $\alpha \in [0, 1]$ is chosen such that $s_\alpha \leq s$ (hence $G(s_\alpha) \subset G(s)$) for all $s \geq 0$ such that $\|s - \bar{s}\|_\infty \leq \delta$ (fig. 1). We set $\alpha := 1 - \delta/\chi \in [0, 1]$ which, given the definition of γ,

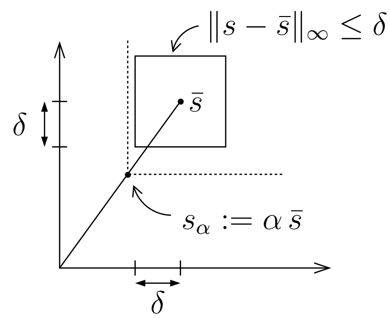

Figure 1: Choice of α

ensures $\alpha \bar{s}_i \leq \bar{s}_i - \delta$ for all i such that $\bar{s}_i > 0$ (the i's such that $\bar{s}_i = 0$ obviously satisfy $\alpha \bar{s}_i \leq s_i$ for all $s \geq 0$, whatever the choice of $\alpha \in [0, 1]$).

Now set $v_\alpha := (1 - \alpha)\, v_0 + \alpha\, \bar{v}$ (which is compatible with the notation v_0). Due to the convexity of graph(G) (Lemma 2.2), $v_\alpha \in G(s_\alpha)$. Hence, for all $s \geq 0$ such that $\|s - \bar{s}\|_\infty \leq \delta$ there holds $v_\alpha \in G(s)$.

Let us sum everything up; we fixed δ, then for all $\bar{v} \in G(\bar{s})$, we constructed v_α which belongs to $G(s)$ for all $s \geq 0$ such that $\|s - \bar{s}\|_\infty \leq \delta$. Moreover, $\bar{v} = v_\alpha + (\bar{v} - v_\alpha)$ with

$$\|\bar{v} - v_\alpha\| = (1 - \alpha)\|v_0 - \bar{v}\| = \frac{\delta}{\chi}\|v_0 - \bar{v}\| \leq \varepsilon$$

which ends the proof. ∎

We conclude with the lemma we will need later on.

Lemma 2.5. *The multi-application G defined by* (2.3) *is uniformly continuous on every compact.*

Proof. By Lemmas 2.2 and 2.4, G is closed and inner semi-continuous, and it is obviously bounded since $G(s) \subset K$ for all $s \in \mathbb{R}_+^{n_I}$. We conclude with Lemma 2.1. ∎

3 Existence of a fixed point

In this section, we prove our existence result, showing that under the assumption (1.20), there exists a fixed point to F. We aim at applying the standard Brouwer's fixed point theorem, so we need two properties: boundedness and continuity of F.

3.1 Boundedness directly

The boundedness of F comes easily from the monotonicity of $\bar{C}$ (Lemma 2.3).

Lemma 3.1 (Boundedness)**.** *Assume* (1.20) *holds, then the function v defined by* (1.17) *and the function F defined by* (1.19) *are bounded for $s \in \mathbb{R}_+^{n_I}$.*

Proof. From Lemma 2.3, there holds $\bar{C}(0) \subset \bar{C}(s)$. Then $J(v(s)) \leq J(v(0)) < +\infty$ so that for all $s \in \mathbb{R}_+^n$, $v(s) \in K$ (recall (2.4)). In other words, the image of v is included in the sub-level set of J at $v(0)$. Since the strong convexity of the quadratic function J implies that its sublevel-sets are bounded, we get that v is bounded. By definition of F, the boundedness of F follows immediately. ∎

3.2 Continuity by perturbation

We prove here the most difficult technical result: the fact that F defined by (1.19) is continuous under (1.20). We will show it by a perturbation argument which allows us to get back the stronger assumption (1.21). The key observation is the following easy result.

Lemma 3.2 (Perturbation of the weak assumption)**.** *Assume that the data of the problem* (1.12) *is such that* (1.20) *holds. For $\delta > 0$, set $\Delta := (\delta, \ldots, \delta) \in \mathbb{R}_+^{n_I}$, and consider the function F_δ defined by* (1.19) *for the data* (1.12) *where w is replaced by $w_\delta = w + E\Delta$. Then the function F_δ is continuous on $\mathbb{R}_+^{n_I}$.*

Proof. Assumption (1.20) (with w) means there exists $v \in \mathbb{R}^m$ such that $Hv + w \in L^*$. Observe now that $E\Delta$ lies in the cone $\mathcal{L}^*$ defined by (1.6). We will show that $Hv + w + E\Delta \in \mathcal{L}^*$ componentwise, by a very similar argument as in the proof of Lemma 2.3.

Since $\Delta \in \mathbb{R}_+^{n_I}$, we see from (1.13) that $(Hv + w + E\Delta)^i \in \mathcal{K}^{*i}$ if $i \notin I$. On the other hand, take $i \in I$; knowing that $e^i \in K^*_{e^i,\mu^i}$ and $\mu^i\delta \geq 0$, we have

$$\mu^i \delta e^i \in K^*_{e^i,\mu^i}.$$

Add to $(Hv + w + Es)^i \in \mathcal{K}^{*i} = \operatorname{int} K^*_{e^i,\mu^i}$ and invoke [HUL93, lemma III.2.1.6]:

$$z := \frac{1}{2}(Hv + w)^i + \frac{1}{2}\mu^i \delta e^i \in \mathcal{K}^{*i}.$$

By positive homogeneity, $(Hv + w + Et)^i = 2z$ also lies in $\mathcal{K}^{*i}$. Thus the assumption (1.21) holds for the data (1.12) with w_δ. We apply Theorem 3.4 to conclude. ∎

We will show that $(F_\delta)_\delta$ converges to F uniformly with respect to s over a closed ball. We start with a lemma to control the difference between v_δ and v.

Lemma 3.3. *Assume that* (1.20) *holds. Let v and v_δ defined by* (1.17) *for* (1.12) *with respectively w and w_δ. Let $D \subset \mathbb{R}_+^{n_I}$ be a compact set.*

$$\forall \varepsilon > 0,\ \exists \bar{\delta} > 0,\ \forall \delta \in [0, \bar{\delta}],\ \forall s \in D:\ \|v_\delta(s) - v(s)\| \leq \varepsilon.$$

Proof. Since J is strongly convex, there exists $\alpha > 0$ such that for all $s \in \mathbb{R}_+^{n_I}$, for all $\delta > 0$ and for all $v \in \mathbb{R}^m$

$$J(v) \geq J(v_\delta(s)) + \nabla J(v_\delta(s))^\top (v - v_\delta(s)) + \alpha \|v - v_\delta(s)\|^2.$$

Let $s_\delta := s + \Delta$. If v lies in $G(s)$ then in $G(s_\delta)$ by Lemma 2.3, we have that the optimality of $v_\delta(s)$ implies $\nabla J(v_\delta(s))^\top (v - v_\delta(s)) \geq 0$. The following growth condition thus holds

$$\eta_s := \alpha \|v_\delta(s) - v(s)\|^2 \leq J(v(s)) - J(v_\delta(s)). \tag{3.1}$$

Furthermore, the quadratic function J has the Lipschitz property over the compact set K for some Lipschitz constant κ. Let $\varepsilon > 0$; set

$$\gamma := \frac{\alpha \varepsilon^2}{\kappa}.$$

The uniform continuity of G over the compact set D (Lemma 2.5) implies the existence of a $\bar{\delta} > 0$ such that, for all $\delta \in [0, \bar{\delta}]$ and for all $s \in D$ such that $s_\delta \in D$, there holds

$$G(s_\delta) \subset G(s) + B(0, \gamma).$$

Since $v_\delta(s) \in G(s_\delta)$, there exists $\omega_\delta \in G(s)$ such that $\|\omega_\delta - v_\delta(s)\| \leq \gamma$. Let us introduce ω_δ in (3.1); we get

$$\eta_s \leq [J(v(s)) - J(\omega_\delta)] + [J(\omega_\delta) - J(v_\delta(s))] \leq J(\omega_\delta) - J(v_\delta(s))$$

since $J(v(s)) \leq J(\omega_\delta)$ by definition of $v(s)$, and finally

$$\eta_s \leq \kappa \|\omega_\delta - v_\delta(s)\| \leq \kappa\gamma$$

using the Lipschitz property. By definition of η_s, this shows that

$$\|v_\delta(s) - v(s)\| \leq \sqrt{\frac{\kappa\gamma}{\alpha}} = \varepsilon$$

which ends the proof. ■

Theorem 3.4 (Continuity of F)**.** *Let $R \geq 0$ that is such that $F(s) \leq R$ for all $s \in \mathbb{R}^{n_I}$ (given by Lemma 3.1). Then F_δ converges to F as $\delta \to 0$ uniformly with respect to $s \in \mathbb{R}_+^{n_I} \cap B(0, R)$. Therefore F is continuous on $\mathbb{R}_+^{n_I} \cap B(0, R)$.*

Proof. For any component i, we have the following inequalities

$$\begin{aligned}
|F_\delta^i(s) - F^i(s)| &= |\|\tilde{u}_{\delta,T}^i\| - \|\tilde{u}_T^i\|| \\
&\leq \|\tilde{u}_{\delta,T}^i - \tilde{u}_T^i\| = \|(\tilde{u}_\delta^i - \tilde{u}^i)_T\| \\
&\leq \|\tilde{u}_\delta^i - \tilde{u}^i\| \\
&\leq \|\tilde{u}_\delta - \tilde{u}\| = \|Hv_\delta(s) + w + E(s + \Delta) - (Hv(s) + w + Es)\| \\
&\leq \|H\| \, \|v_\delta(s) - v(s)\| + \|E\| \, \|\Delta\| = \|H\| \|v_\delta(s) - v(s)\| + \|E\| \delta \sqrt{n}
\end{aligned}$$

Since $v_\delta(\cdot)$ converges to $v(\cdot)$ uniformly with respect to s (with $\|s\| \leq R$) as δ goes to zero (by Lemma 3.3), we have that F_δ converges uniformly to F. Lemma 3.2 proves that the functions F_δ are continuous and this concludes the proof. ■

3.3 Proof of the result

We are now in position to state the existence of a fixed point to F under the assumption (1.20): the proof follows easily from the gathered previous results.

Theorem 3.5 (Existence of fixed-point)**.** *If there exists $v \in \mathbb{R}^m$ such that $Hv + w$ lies in L^*, then F defined by* (1.19) *admits a fixed point on $\mathbb{R}_+^{n_I}$.*

Proof. The function F is nonnegative and Lemma 3.1 shows that it is bounded. We introduce $R \geq 0$ such that $\|F(x)\| \leq R$, and thus we have

$$F(\mathbb{R}_+^{n_I} \cap B(0,R)) \subset \mathbb{R}_+^{n_I} \cap B(0,R).$$

Theorem 3.4 gives that F is continuous on $\mathbb{R}_+^n \cap B(0,R)$. So we can apply the Brouwer's fixed point theorem (see e.g. [Ist81] or [DS88]) to F on $\mathbb{R}_+^n \cap B(0,R)$ and we obtain the existence of at least one fixed point of F on $\mathbb{R}_+^n \cap B(0,R)$. ■

This result thus generalizes the existence result of [ACML10] under the weaker, natural assumption (1.20). The key of the proof is the continuity property (Lemma 2.5) of the set-valued mapping corresponding the constraint set of the optimization problem (1.14) used to defined the function F. Numerical experiments and mechanical interpretation are developped in [Cad09] and [ACML10].

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